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EXPECTED NUMBER OF MAXIMA AND MINIMA OF A STATIONARY RANDOM PROCESS WITH NON-GAUSSIAN FREQUENCY DISTRIBUTION

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SUMMARY

A method is outlined for calculating the expected number of maxima or minima of a random process with non-Gaussian frequency distribution from the statistical moments of the process and its first two derivatives. This method is based on an estimate of the joint frequency function of the process and its first two derivatives given by means of a generalized form of Edgeworth's series; the procedure thus consists essentially in applying a correction to the results for a Gaussian process. The functions required in this procedure are calculated for the first two correction terms; therefore, the effects of skewness and kurtosis can be calculated, provided the required moments are known. Expressions are given for these moments in terms of multiple correlation functions and multispectra, and the relations between these functions for a random output of a linear system and those for the random input are indicated.

INTRODUCTION

Many physical processes of interest in aeronautics and allied fields are determinate only in a statistical sense. Such processes are referred to as stochastic or random processes. If the statistical characteristics of such a process are invariant in time, it is referred to as a stationary random process. The basic problem in connection with these processes is usually either to predict the output of a dynamic system which is subjected to a random input (so that the output is also generally random in nature) from the statistical characteristics of the input and the dynamic characteristics of the system, or to estimate certain statistical characteristics of a given process from others. (See refs. 1 to 6 for discussions of several problems in communications theory and aeronautics from the point of view of random-process theory.)

One statistical characteristic which is frequently of interest is the number of maxima or minima expected in a given time; that is, the number of positive or negative peaks of the process within a certain range or exceeding a certain level that can be expected in that time. The expected life of an airplane, for instance, depends on the expected number of times in a given period of time that its ultimate load is likely to be exceeded. (See refs. 4 and 5.) Similarly, the fatigue life of a structure can in some cases be related to the number of maxima per unit time and their frequency distribution. (See ref. 6, for instance.)

For a stationary Gaussian process - that is, for a stationary random process in which the stochastic variable and its derivative are jointly normally distributed - Rice (ref. 1) has given a simple expression for the expected number of maxima in terms of the second moments of the process and its first two derivatives. These moments can be obtained from the correlation function or power spectrum of the process. In turn, if the process represents the output of a linear system, the spectrum can be related very simply to the correlation function or spectrum of the input.

In the present paper a similar expression for the expected number of maxima is obtained for a stationary process with a joint frequency distribution of the process and its first two derivatives; this distribution differs slightly from the normal. The approach used herein consists in expressing the joint frequency distribution of the process and its derivatives in terms of its second and higher moments by means of a multivariate form of Edgeworth's series, so that the desired expected number of maxima or minima can then be expressed in terms of these moments. Again, these moments can be expressed in terms of correlation functions and spectra, and the correlation functions and spectra of an output can be related readily to those of the input. However, in this case more than the ordinary (double) correlation function or power spectrum is required, because the nth moments depend on the n-tuple correlation function or the corresponding spectrum. Hence, some of the multiple correlation functions or multispectra of the input must be known if the number of maxima of an output process with non-Gaussian frequency distribution is to be predicted by this method.

Inasmuch as the terms of Edgeworth's series represent, essentially, corrections to a normal distribution, the approach outlined herein also furnishes, essentially, a correction to the results obtained for a Gaussian process. Explicit expressions are given herein for the functions required in the first two correction terms, which involve the third and fourth moments. No such expressions are given for higher correction terms, because the effort entailed in obtaining the required multispectra soon becomes very large. The procedure given herein furnishes an estimate of the effects of skewness and kurtosis on the results of interest and is, therefore, best suited to distributions which differ by relatively little from the normal one, that is, primarily in the third and fourth moments but to a lesser extent in the higher moments.

Consideration will be confined to a process with zero mean. This trivial restriction implies that, for a process with nonzero mean, the results given here apply directly to the process which is the difference between the actual process and its mean; they can, however, be modified to apply to the actual process in a straightforward fashion.

SYMBOLS

D^n	operator $\frac{\partial^n}{\partial t^n}$
$f(x_0,x_1,x_2)$	trivariate frequency-distribution function for a random process and its first two derivatives
$f_G(x_0,x_1,x_2)$	trivariate normal distribution (with zero means)
$F(t)$	random input process
$h(t)$	indicial-response function for linear dynamic system
$H(\omega)$	frequency-response function for linear dynamic system
$I_{mnp}(x_0)$	function used in expression for $n(x_0)$, defined in equation (17)
$J_{mnp}(x_0)$	function used in expression for $N(x_0)$, defined in equation (21)
M_{ij}	reciprocal second moments (elements of inverse of matrix of second moments)
$n(x_0)$	expected number per unit time of maxima with intensities in band of unit width centered on x_0
$N(x_0)$	expected number per unit time of maxima above x_0
t	time
$x(t)$	given random process, which may be output of a linear system subject to random input
$\dot{x}(t),\ddot{x}(t)$	first and second time derivatives of $x(t)$

x_0, x_1, x_2	random variables corresponding to x, $\dot{x}$, and $\ddot{x}$, respectively
α	parameter, $\sqrt{\Delta}\Big/\overline{\dot{x}^2}^{3/2}$
α_{mnp}	joint statistical moments of random process and its first two derivatives, defined in equation (6a) or (6b)
α^{ijk}	joint statistical moments of random process and its first two derivatives, defined in equation (5)
Δ	determinant of matrix of second moments
ξ	dimensionless random variable, $\frac{1}{\alpha}\frac{x_0}{\sqrt{\overline{x^2}}}$
ψ_x	(double) correlation function for $x(t)$
ψ_{xx}	triple correlation function for $x(t)$
ψ_{xxx}	quadruple correlation function for $x(t)$
φ_x	power spectrum for $x(t)$, Fourier transform of ψ_x
φ_{xx}	double power spectrum for $x(t)$, Fourier transform of ψ_{xx}
φ_{xxx}	triple power spectrum for $x(t)$, Fourier transform of ψ_{xxx}
τ	time displacement, argument of ψ_x
ω	circular frequency

ANALYSIS

Basic Relations

For a given stationary random process $x(t)$, the number of maxima that are expected to exceed the level x_0 per unit time will be designated by $N(x_0)$, and the number of maxima in an intensity band of unit width centered on x_0 that are likely to occur per unit time will be designated by $n(x_0)$, so that

$$n(x_0) = -\frac{dN(x_0)}{dx_0} \tag{1}$$

In reference 1 the following expression is given for $n(x_0)$ in terms of the joint frequency distribution $f(x_0,x_1,x_2)$ of the process and its first two derivatives:

$$n(x_0) = \int_{-\infty}^{0} |x_2| f(x_0,0,x_2)\,dx_2 \tag{2}$$

where $f(x_0,x_1,x_2)$ is defined by the fact that $f(x_0,x_1,x_2)\,dx_0 dx_1 dx_2$ represents the probability that at time t:

$$x_0 \leqq x(t) < x_0 + dx_0$$

$$x_1 \leqq \dot{x}(t) < x_1 + dx_1$$

$$x_2 \leqq \ddot{x}(t) < x_2 + dx_2$$

This function is invariant with t by virtue of the assumed stationarity of the process.

For a Gaussian process this result can be expressed in an especially simple form. For such a process the frequency function is

$$f_G(x_0,x_1,x_2) = \frac{1}{(2\pi)^{3/2}\sqrt{\Delta}} e^{-\frac{1}{2}\left(M_{00}x_0^2+M_{11}x_1^2+M_{22}x_2^2+2M_{02}x_0x_2\right)} \tag{3}$$

where Δ is the determinant of the second moments of the process, and the coefficients M_{ij} are the elements of a matrix reciprocal to the matrix of the second moments and will be discussed further in a later section. Substitution of this expression for $f(x_0,x_1,x_2)$ in equation (2) yields an expression for $n(x_0)$ in terms of the moments; hence, integration over x_0 yields an expression for $N(x_0)$, and thus also for total number of maxima per unit time, which is equal to $N(-\infty)$. The expressions for $n(x_0)$ and $N(-\infty)$ are given in reference 1.

For a non-Gaussian process a similar expression for the joint frequency distribution in terms of the moments can be obtained from a multivariate version of Edgeworth's series. This series is derived (see ref. 7, for instance) on the assumption that the given process represents the sum of a large number M of statistically independent random variables. Then, by expanding the characteristic function for the process in a series, several asymptotic expressions can be obtained (depending on the manner in which the terms are collected) for the given non-Gaussian distribution in terms of its moments, the Gaussian distribution, and its derivatives. As M tends toward infinity, all terms of the series except the one which represents the Gaussian part of the distribution tend to zero. In Edgeworth's series, terms are grouped according to powers of M, so that each group can be expected to contain the terms representing a given extent of deviation from the Gaussian distribution.

This derivation can readily be extended to multivariate distributions by using the concepts of random-vector theory, such as the multivariate form of the characteristic function and of the Gaussian distribution. (See ref. 7, for instance.) The results can be expressed for the case of interest as

$$f(x_0,x_1,x_2) = \left[1 - \left\{\frac{\alpha^{ijk}}{3!}\frac{\partial^3}{\partial x_i \partial x_j \partial x_k}\right\} + \left\{\frac{\alpha^{ijkl} - 3\alpha^{ij}\alpha^{kl}}{4!}\frac{\partial^4}{\partial x_i \partial x_j \partial x_k \partial x_l} + \frac{10\alpha^{ijk}\alpha^{lmn}}{6!}\frac{\partial^6}{\partial x_i \partial x_j \partial x_k \partial x_l \partial x_m \partial x_n}\right\} - \cdots\right] f_G(x_0,x_1,x_2) \tag{4}$$

where $\alpha^{ijkl\ldots}$ represents the moments of the given process and the indices $i, j, k, \ldots$ may have the value 0, 1, or 2. In this expression the summation convention is used, so that any index repeated in a product implies a summation over that index. Terms associated (in the derivation) with a given power of M are grouped within braces. Only the first two terms beyond the Gaussian part are listed here; from the Edgeworth's series given in reference 7, one additional term can be deduced by analogy.

The Statistical Moments of the Process

As used in equation (4), the moments are defined by

$$\alpha^{ijk\ldots} \equiv \overline{D^i\left\{x(t)\right\}D^j\left\{x(t)\right\}D^k\left\{x(t)\right\}\ldots} \tag{5}$$

where the bar designates a time average, and the symbols D^i, D^j, . . . designate the ith, jth, . . . derivative with respect to time. The reason for this definition is that it permits the application of the summation convention and thus greatly simplifies the writing of equation (4), and the reason for the superscript notation is that subscripts will be used for the moments defined in the manner which is more descriptive and convenient for the purpose at hand, because it assigns only one set of indices to any moment, namely

$$\alpha_{mnp} \equiv \overline{x^m(t)\dot{x}^n(t)\ddot{x}^p(t)} \tag{6a}$$

or

$$\alpha_{mnp} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_0{}^m x_1{}^n x_2{}^p f\left(x_0,x_1,x_2\right)dx_0\ dx_1\ dx_2 \tag{6b}$$

The two sets of moments defined by equations (5) and (6) can be identified with each other in the following manner: For an rth moment the number of superscripts $i, j, k, \ldots$ is r, and $m + n + p = r$; m, n, and p are, respectively, the numbers of 0's, 1's, and 2's among $i, j, k, \ldots$. Thus, for instance,

$$\overline{x^2\ \dot{x}\ \ddot{x}^3} \equiv \alpha_{213} \equiv \overline{x\ x\ \dot{x}\ \ddot{x}\ \ddot{x}\ \ddot{x}} \equiv \alpha^{001222}$$

$$= \alpha^{222100} = \alpha^{120220}$$

and so on. Obviously, the superscripts of $\alpha^{ijk\ldots}$ can be permuted in any manner, but any change in the subscripts of α_{mnp} changes the moment referred to.

The first moments (mean values) of the process $\alpha_{100} = \overline{x}$, $\alpha_{010} = \overline{\dot{x}}$, and $\alpha_{001} = \overline{\ddot{x}}$ are zero; $\overline{x}$ is zero by stipulation and the others are zero of necessity, inasmuch as the process is stationary. Of the nine second moments, four are zero and two are equal to each other as a result of the assumed stationarity of the process; therefore, the following moments are left:

$$\alpha_{200} = \overline{x^2}$$

$$\alpha_{020} = \overline{\dot{x}^2}$$

$$\alpha_{002} = \overline{\ddot{x}^2}$$

$$\alpha_{101} = \overline{x\ddot{x}}$$

As pointed out in reference 7, for instance, they can all be expressed in terms of the (double) correlation function for $x(t)$, which is defined by

$$\psi_x(\tau) = \overline{x(t)x(t+\tau)} \tag{7}$$

or in terms of the power spectrum, which is defined by

$$\varphi_x(\omega) \equiv \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-i\omega\tau}\psi_x(\tau)d\tau \tag{8}$$

The required relations are given in reference 7 and are repeated for the sake of completeness in table 1.

The second moments can be arranged in a matrix in the following manner:

$$\left[\alpha^{ij}\right] \equiv \begin{bmatrix} \alpha_{200} & 0 & -\alpha_{020} \\ 0 & \alpha_{020} & 0 \\ -\alpha_{020} & 0 & \alpha_{002} \end{bmatrix} = \begin{bmatrix} \overline{x^2} & 0 & -\overline{\dot{x}^2} \\ 0 & \overline{\dot{x}^2} & 0 \\ -\overline{\dot{x}^2} & 0 & \overline{\ddot{x}^2} \end{bmatrix}$$

(in which the fact was used that $\alpha_{101} = -\alpha_{020}$, as may be noted from table 1). The determinant of this matrix is

$$\Delta \equiv \alpha_{020}\left(\alpha_{200}\alpha_{002} - \alpha_{020}^{2}\right) \tag{9}$$

and the inverse of this matrix is

$$\left[M_{ij}\right] \equiv \begin{bmatrix} M_{00} & M_{01} & M_{02} \\ M_{10} & M_{11} & M_{12} \\ M_{20} & M_{21} & M_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{\alpha_{002}}{\alpha_{200}\alpha_{002} - \alpha_{020}^{2}} & 0 & \dfrac{\alpha_{020}}{\alpha_{200}\alpha_{002} - \alpha_{020}^{2}} \\ 0 & \dfrac{1}{\alpha_{020}} & 0 \\ \dfrac{\alpha_{020}}{\alpha_{200}\alpha_{002} - \alpha_{020}^{2}} & 0 & \dfrac{\alpha_{200}}{\alpha_{200}\alpha_{002} - \alpha_{020}^{2}} \end{bmatrix} \tag{10}$$

Both the determinant and the nonzero elements of M_{ij} occur in equation (3).

Of the 27 third moments only 10 are distinct, and two of those are zero as a result of stationarity. The remaining 8 are listed in table 1, and expressions are given for them in terms of the triple correlation function and double spectrum defined by

$$\psi_{xx}(\tau_1,\tau_2) \equiv \overline{x(t)x\left(t+\tau_1\right)x\left(t+\tau_2\right)} \tag{11}$$

and

$$\varphi_{xx}(\omega_1,\omega_2) = \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\left(\omega_1\tau_1+\omega_2\tau_2\right)}\psi_{xx}\left(\tau_1,\tau_2\right)d\tau_1\, d\tau_2 \tag{12}$$

These expressions can be derived readily from the definitions of the moments, of the correlation function, and of the spectrum by a straightforward extension of the procedure used for the second moments; these derivations were obtained by using integrations by parts, differentiations under the integral sign, and similar elementary operations, and by taking advantage of the fact that the process is stationary. Some of these moments have been calculated in reference 8 for the purpose of estimating the frequency distribution of $x(t)$ alone.

Similarly, of the 64 fourth moments the 13 distinct and nonzero ones are given in table 1, as are expressions for them in terms of the quadruple correlation function and triple spectrum

$$\psi_{xxx}(\tau_1,\tau_2,\tau_3) \equiv \overline{x(t)x(t+\tau_1)x(t+\tau_2)x(t+\tau_3)} \tag{13}$$

and

$$\varphi_{xxx}(\omega_1,\omega_2,\omega_3) \equiv \frac{1}{\pi^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\omega_1\tau_1+\omega_2\tau_2+\omega_3\tau_3)}\,\psi_{xxx}(\tau_1,\tau_2,\tau_3)\,d\tau_1\,d\tau_2\,d\tau_3 \tag{14}$$

Expressions for the Expected Number of Maxima

If the expression for the joint frequency distribution given by equation (4) is substituted into equation (2), the following expression is obtained for $n(x_0)$:

$$n(x_0) = \frac{1}{(2\pi)^{3/2}\sqrt{\Delta}}\left[I - \left\{\frac{\alpha^{ijk}}{3!}\,I^{ijk}(x_0)\right\} + \left\{\frac{\alpha^{ijkl} - \alpha^{ij}\alpha^{kl}}{4!}\,I^{ijkl}(x_0) + \frac{10\alpha^{ijk}\alpha^{lmn}}{6!}\,I^{ijklmn}(x_0)\right\} - \ldots\right] \tag{15}$$

where

$$I^{ijk\cdots}(x_0) \equiv \int_{-\infty}^{0}|x_2|\frac{\partial^r}{\partial x_i\partial x_j\partial x_k\ldots}\,f_G(x_0,0,x_2)\,(2\pi)^{3/2}\sqrt{\Delta}\,dx_2 \tag{16}$$

where r is the number of indices $i, j, k, \ldots$.

As in the case of the moments, another definition of these functions is more convenient for some purposes, namely,

$$I_{mnp}(x_0) \equiv \int_{-\infty}^{0} |x_2| \frac{\partial^{m+n+p}}{\partial x_0{}^m \partial x_1{}^n \partial x_2{}^p} f_G(x_0,0,x_2)(2\pi)^{3/2}\sqrt{\Delta}\, dx_2 \tag{17}$$

These functions can be identified with those defined by equation (16) in the manner indicated for the moments defined by equations (5) and (6), respectively.

These functions are listed in table 2, in terms of the dimensionless variable

$$\xi \equiv \frac{1}{\alpha} \frac{x_0}{\sqrt{\overline{x^2}}} \tag{18}$$

and the parameter

$$\alpha \equiv \frac{\sqrt{\Delta}}{\overline{\dot{x}^2}^{3/2}} \tag{19}$$

The functions $E(z)$ and $E^*(z)$ in this table are defined by

$$E(z) \equiv \int_{-\infty}^{z} e^{-\frac{\xi^2}{2}} d\xi$$

$$= \sqrt{\frac{\pi}{2}}\left[1 + \operatorname{erf}\left(\frac{z}{\sqrt{2}}\right)\right]$$

and

$$E^*(z) \equiv \int_{z}^{\infty} e^{-\frac{\xi^2}{2}} d\xi$$

$$= \sqrt{\frac{\pi}{2}}\left[1 - \operatorname{erf}\left(\frac{z}{\sqrt{2}}\right)\right]$$

$$= \sqrt{\frac{\pi}{2}} \operatorname{erfc}\left(\frac{z}{\sqrt{2}}\right)$$

where erf and erfc designate the error function and the complementary error function, respectively. The functions not listed in table 2 are zero or not required because they are multiplied by a moment which is zero. Also included in table 2 for the sake of completeness is the function for Gaussian distributions I_{000}.

In terms of these functions, equation (15) can be written as:

$$
\begin{aligned}
(2\pi)^{3/2}\sqrt{\Delta}\, n(x_0) = I_{000} - &\left\{\frac{1}{3!}\left[\alpha_{300}I_{300} + 3\alpha_{201}I_{201} + 3\alpha_{120}I_{120} + 3\alpha_{102}I_{102} + \alpha_{003}I_{003}\right]\right\} + \\
&\left\{\frac{1}{4!}\left[\left(\alpha_{400} - 3\alpha_{200}^2\right)I_{400} + 4\left(\alpha_{301} + 3\alpha_{200}\alpha_{020}\right)I_{301} + 6\left(\alpha_{220} - \alpha_{200}\alpha_{020}\right)I_{220} + \right.\right. \\
&6\left(\alpha_{202} - 2\alpha_{020}^2 - \alpha_{200}\alpha_{002}\right)I_{202} + 12\left(\alpha_{121} + \alpha_{020}^2\right)I_{121} + 4\left(\alpha_{103} + 3\alpha_{020}\alpha_{002}\right)I_{103} + \\
&\left.\left(\alpha_{040} - 3\alpha_{020}^2\right)I_{040} + 6\left(\alpha_{022} - \alpha_{020}\alpha_{002}\right)I_{022} + \left(\alpha_{004} - 3\alpha_{002}^2\right)I_{004}\right] + \\
&\frac{10}{6!}\left[\alpha_{300}^2 I_{600} + 6\alpha_{300}\alpha_{201}I_{501} + 6\alpha_{300}\alpha_{120}I_{420} + \left(6\alpha_{300}\alpha_{102} + 9\alpha_{201}^2\right)I_{402} + \right. \\
&18\alpha_{201}\alpha_{120}I_{321} + \left(2\alpha_{300}\alpha_{003} + 18\alpha_{102}\alpha_{201}\right)I_{303} + 9\alpha_{120}^2 I_{240} + \left(18\alpha_{120}\alpha_{102} + \right. \\
&\left.36\alpha_{111}^2\right)I_{222} + \left(6\alpha_{201}\alpha_{003} + 9\alpha_{102}^2\right)I_{204} + 12\alpha_{030}\alpha_{111}I_{141} + \left(6\alpha_{003}\alpha_{120} + \right. \\
&\left.36\alpha_{012}\alpha_{111}\right)I_{123} + 6\alpha_{102}\alpha_{003}I_{105} + \alpha_{030}^2 I_{060} + 6\alpha_{030}\alpha_{012}I_{042} + \\
&\left.\left.9\alpha_{012}^2 I_{024} + \alpha_{003}^2 I_{006}\right]\right\} - \cdot\cdot\cdot
\end{aligned}
\tag{20}
$$

A corresponding expression for the expected maxima per unit time exceeding the level x_0 can be obtained by integrating this expression over x_0, inasmuch as

$$
N(x_0) = \int_{x_0}^{\infty} n(x_0')\, dx_0'
$$

(see eq. (1)). The resulting expression for $N(x_0)$ has a form identical to equation (20) but with all functions $I_{mnp}(x_0)$ replaced by functions $J_{mnp}(x_0)$ defined by

$$
J_{mnp}(x_0) \equiv \int_{x_0}^{\infty} I_{mnp}(x_0')\, dx_0' \tag{21}
$$

These functions are also given in table 2. Finally, an expression for the total number of maxima per unit time can be obtained by replacing $I_{mnp}(x_0)$

in equation (20) by $J_{mnp}(-\infty)$. For the sake of convenience the values of $J_{mnp}(-\infty)$ are also listed in table 2.

Input-Output Relations

If $x(t)$ is the output of a linear system subjected to an input $F(t)$, the double and higher order correlation functions and spectra of $x(t)$, which are required to obtain the moments when no other information is available, can be related to the correspondingly defined function for $F(t)$. In this process either the indicial response $h(t)$, that is, the response to an impulsive input, or the transfer function $H(\omega)$, that is, the complex amplitude response to steady-state sinusoidal oscillations of unit amplitude, must be known. These two characteristics of the system are related by

$$H(\omega) = \int_0^\infty e^{-i\omega t}h(t)dt \tag{22}$$

The double correlation functions are related by the expression

$$\psi_x(\tau) = \int_{-\infty}^\infty \psi_F(\tau-\sigma)\psi_h(\sigma)d\sigma \tag{23}$$

where

$$\psi_h(\sigma) = \int_0^\infty h(t)h(t+|\sigma|)dt \tag{24}$$

and the corresponding spectra are related more simply by

$$\varphi_x(\omega) = |H(\omega)|^2\varphi_F(\omega) \tag{25}$$

where the vertical bars on $H(\omega)$ designate the absolute value.

Similar relations can readily be derived for the higher order correlation functions and spectra, the expressions for the spectra being

$$\varphi_{xx}(\omega_1,\omega_2) = H(\omega_1)H(\omega_2)H^*(\omega_1+\omega_2)\varphi_{FF}(\omega_1,\omega_2) \tag{26}$$

$$\varphi_{xxx}(\omega_1,\omega_2,\omega_3) = H(\omega_1)H(\omega_2)H(\omega_3)H^*(\omega_1+\omega_2+\omega_3)\varphi_{FFF}(\omega_1,\omega_2,\omega_3) \tag{27}$$

and so on, where the asterisk designates the complex conjugate.

DISCUSSION

In the preceding presentation, attention has been confined to maxima. However, the results can readily be modified to apply to minima as well, because the expected number of minima per unit time in a band of unit width centered on x_0 is given by

$$n^-(x_0) = \int_0^\infty x_2 f(x_0,0,x_2)dx_2 \tag{28}$$

A comparison of this equation with equation (1), which may be written as

$$n(x_0) = \int_0^\infty x_2 f(x_0,0,-x_2)dx_2 \tag{29}$$

indicates that the expected number of minima can be obtained for the expected number of maxima by changing the sign of x_2 in the frequency-distribution function.

Consequently, as a result of the definition of $I_{mnp}(x_0)$ and $J_{mnp}(x_0)$, the expected number of minima can be obtained by making the following two changes in the results presented in this paper for maxima:

(1) Replace M_{02} by $\left(-M_{02}\right)$ wherever it occurs in table 2.

(2) Multiply those of the functions $I_{mnp}(x_0)$ and $J_{mnp}(x_0)$ for which n is odd by (-1).

Similarly, if the number of maxima per unit time <u>below</u> x_0 is desired, this number can be obtained by subtracting $N(x_0)$ from $N(-\infty)$; and in view of the statements made in the preceding paragraph, the same procedure can be used for the number of minima below x_0 if $N(x_0)$ is calculated for minima by changing the sign of M_{02} and of the specified functions. The total number $N(-\infty)$ of maxima is also the number of minima, because for each maximum there must be a minimum.

As may be gathered from the procedure outline herein, when the joint frequency-distribution function is not Gaussian, the computational effort involved in obtaining the desired expected number of maxima or minima soon becomes quite extensive, particularly in view of the fact that usually multiple correlation functions or spectra have to be calculated first in order to obtain the required moments. Although equipment exists to measure such functions directly or through analog-computing devices (see ref. 9, for instance), this equipment has not reached the perfection of the equipment used for the measurement of ordinary correlations and power spectra.

The numerical calculations of these functions from time histories also pose difficulties beyond those resulting from the greater number of variables involved. The source of these difficulties may be described by expressing the power spectrum corresponding to the nth correlation function associated with a given process $x(t)$ in terms of the Fourier transform of the process

$$a(\omega;T) \equiv \int_{-T}^{T} e^{-i\omega t}x(t)dt \tag{30}$$

as

$$\varphi_{xxx\ldots}(\omega_1,\omega_2,\ldots\omega_{n-1}) = \pi \lim_{T\to\infty} \frac{a(\omega_1;T)a(\omega_2;T)\ .\ .\ .\ a(\omega_{n-1};T)a^*(\omega_1+\omega_2+\ldots+\omega_{n-1};T)}{2T} \tag{31}$$

where the asterisk designates the complex conjugate. Hence, in the ordinary spectrum $(n = 2)$ only the absolute magnitude of $a(\omega;T)$ occurs, whereas in all higher spectra the phase enters as well. (See also ref. 8.) Also, inasmuch as the $\lim_{T\to\infty} a(\omega;T)$ does not exist, the existence of the spectra stems from the fact that the product of the functions $a(\omega;T)$ is divided by T before the limit is taken. Consequently, the higher the spectrum the more the effect of the single T in the nominator is "diluted." Consequently, in a practical case, for a given reliability, a greater portion of a random process must be analyzed if higher-order correlation functions or spectra are to be obtained than if only the ordinary functions are of interest. (See also ref. 10.) However, if the deviation from a Gaussian distribution is not too large, the higher order moments are likely to be small compared with the second moments; therefore, a lower degree of reliability in their determination may be acceptable.

In view of these difficulties the procedure outlined herein is likely to find application primarily in problems where the expected number of maxima of an output have to be estimated from knowledge concerning the input, because then even the statistical characteristics of the process are not known. For a given process the procedure outlined herein is likely to become of advantage only once the equipment for measuring higher order correlation functions and spectra is perfected, because if the process itself is known (in the form of a time history) the expected number of maxima can be established by direct count more readily than indirectly through a calculation of the joint frequency distribution or moments; and if the frequency function is known, a direct calculation of the extrema from equation (2) or (28) is likely to be more convenient than an indirect calculation based on moments calculated from this function by using equation (6b).

CONCLUDING REMARKS

A method has been outlined for calculating the expected number of maxima or minima of a random process with non-Gaussian frequency distribution from the statistical moments of the process and its first two derivatives. This method is based on an estimate of the joint frequency function of the process and its first two derivatives by means of a generalized form of Edgeworth's series; the procedure consists essentially in applying a correction to the results for a Gaussian process. The functions required in this procedure have been calculated for the first two correction terms; therefore the effects of skewness and kurtosis can be calculated, provided the third and fourth moments of the process and its first two derivatives are known.

Expressions have been given for these moments in terms of multiple correlation functions and multispectra, and the relations between these functions for the random output of a linear system and those for the random input have been indicated.

Langley Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., January 19, 1957.

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TABLE 1.- EXPRESSIONS FOR THE MOMENTS IN TERMS OF CORRELATION FUNCTIONS AND SPECTRA

Moments		Expressions in terms of $\psi_x(\tau)$, $\psi_{xx}(\tau_1,\tau_2)$, and $\psi_{xxx}(\tau_1,\tau_2,\tau_3)$ (a)	Expressions in terms of $\varphi_x(\omega)$, $\varphi_{xx}(\omega_1,\omega_2)$, and $\varphi_{xxx}(\omega_1,\omega_2,\omega_3)$
Second	α_{200}	ψ_x	$\int_0^\infty \varphi_x \, d\omega$
	α_{101}	$\frac{\partial^2\psi_x}{\partial\tau^2} = -\alpha_{020}$	$-\alpha_{020}$
	α_{020}	$-\frac{\partial^2\psi_x}{\partial\tau^2}$	$\int_0^\infty \varphi_x\omega^2 d\omega$
	α_{002}	$\frac{\partial^4\psi_x}{\partial\tau^4}$	$\int_0^\infty \varphi_x\omega^4 d\omega$
Third	α_{300}	ψ_{xx}	$\frac{1}{4}\int_{-\infty}^\infty\int_{-\infty}^\infty \varphi_{xx} \, d\omega_1 \, d\omega_2$
	α_{201}	$\frac{\partial^2\psi_{xx}}{\partial\tau_1^2}$	$-\frac{1}{4}\int_{-\infty}^\infty\int_{-\infty}^\infty \varphi_{xx}\omega_1^2 d\omega_1 \, d\omega_2$
	α_{120}	$\frac{\partial^2\psi_{xx}}{\partial\tau_1\partial\tau_2} = -\frac{1}{2}\alpha_{201}$	$-\frac{1}{2}\alpha_{201}$
	α_{111}	$\frac{\partial^3\psi_{xx}}{\partial\tau_1\partial\tau_2^2}$	$-\frac{1}{4}\int_{-\infty}^\infty\int_{-\infty}^\infty \varphi_{xx}\omega_1^2\omega_2 \, d\omega_1 \, d\omega_2$
	α_{102}	$\frac{\partial^4\psi_{xx}}{\partial\tau_1^2\partial\tau_2^2}$	$\frac{1}{4}\int_{-\infty}^\infty\int_{-\infty}^\infty \varphi_{xx}\omega_1^2\omega_2^2 d\omega_1 \, d\omega_2$
	α_{030}	$-2\frac{\partial^3\psi_{xx}}{\partial\tau_1\partial\tau_2^2} = -2\alpha_{111}$	$-2\alpha_{111}$
	α_{012}	$-2\frac{\partial^5\psi_{xx}}{\partial\tau_1^2\partial\tau_2^3}$	$-\frac{1}{2}\int_{-\infty}^\infty\int_{-\infty}^\infty \varphi_{xx}\omega_1^3\omega_2^2 \, d\omega_1 \, d\omega_2$
	α_{003}	$2\left(\frac{\partial^6\psi_{xx}}{\partial\tau_1^3\partial\tau_2^3} + \frac{\partial^6\psi_{xx}}{\partial\tau_1^4\partial\tau_2^2}\right)$	$-\frac{1}{2}\int_{-\infty}^\infty\int_{-\infty}^\infty \varphi_{xx}\left(\omega_1^3\omega_2^3+\omega_1^4\omega_2^2\right)d\omega_1 \, d\omega_2$

[a] The functions $\psi_x(\tau)$, $\psi_{xx}(\tau_1,\tau_2)$, and $\psi_{xxx}(\tau_1,\tau_2,\tau_3)$ and their derivatives are to be evaluated at $\tau = \tau_1 = \tau_2 = \tau_3 = 0$.

TABLE 1.- EXPRESSIONS FOR THE MOMENTS IN TERMS OF CORRELATION FUNCTIONS AND SPECTRA - Concluded

Moments		Expressions in terms of $\psi_x(\tau)$, $\psi_{xx}(\tau_1,\tau_2)$, and $\psi_{xxx}(\tau_1,\tau_2,\tau_3)$ (a)	Expressions in terms of $\varphi_x(\omega)$, $\varphi_{xx}(\omega_1,\omega_2)$, and $\varphi_{xxx}(\omega_1,\omega_2,\omega_3)$
Fourth	α_{400}	ψ_{xxx}	$\frac{1}{8}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi_{xxx}\,d\omega_1\,d\omega_2\,d\omega_3$
	α_{301}	$\frac{\partial^2\psi_{xxx}}{\partial\tau_1^2}$	$-\frac{1}{8}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi_{xxx}\omega_1^2\,d\omega_1\,d\omega_2\,d\omega_3$
	α_{220}	$\frac{\partial^2\psi_{xxx}}{\partial\tau_1\partial\tau_2} = -\frac{1}{3}\alpha_{301}$	$-\frac{1}{3}\alpha_{301}$
	α_{211}	$\frac{\partial^3\psi_{xxx}}{\partial\tau_1^2\partial\tau_2}$	$-\frac{1}{8}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi_{xxx}\omega_1^2\omega_2\,d\omega_1\,d\omega_2\,d\omega_3$
	α_{202}	$\frac{\partial^4\psi_{xxx}}{\partial\tau_1^2\partial\tau_2^2}$	$\frac{1}{8}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi_{xxx}\omega_1^2\omega_2^2\,d\omega_1\,d\omega_2\,d\omega_3$
	α_{130}	$\frac{\partial^3\psi_{xxx}}{\partial\tau_1\partial\tau_2\partial\tau_3} = -\alpha_{211}$	$-\alpha_{211}$
	α_{121}	$\frac{\partial^4\psi_{xxx}}{\partial\tau_1^2\partial\tau_2\partial\tau_3} = \frac{1}{6}\frac{\partial^4\psi_{xxx}}{\partial\tau_1^4} - \frac{1}{2}\alpha_{202}$	$\frac{1}{8}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi_{xxx}\omega_1^2\omega_2\omega_3\,d\omega_1\,d\omega_2\,d\omega_3$
	α_{112}	$\frac{\partial^5\psi_{xxx}}{\partial\tau_1^2\partial\tau_2^2\partial\tau_3}$	$\frac{1}{8}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi_{xxx}\omega_1^2\omega_2^2\omega_3\,d\omega_1\,d\omega_2\,d\omega_3$
	α_{103}	$\frac{\partial^6\psi_{xxx}}{\partial\tau_1^2\partial\tau_2^2\partial\tau_3^2}$	$-\frac{1}{8}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi_{xxx}\omega_1^2\omega_2^2\omega_3^2\,d\omega_1\,d\omega_2\,d\omega_3$
	α_{040}	$-3\frac{\partial^4\psi_{xxx}}{\partial\tau_1^2\partial\tau_2\partial\tau_3} = -3\alpha_{121}$	$-3\alpha_{121}$
	α_{022}	$-\left(\frac{\partial^6\psi_{xxx}}{\partial\tau_1^2\partial\tau_2^2\partial\tau_3^2} + 2\frac{\partial^6\psi_{xxx}}{\partial\tau_1^3\partial\tau_2^2\partial\tau_3}\right)$	$\frac{1}{8}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi_{xxx}\left(\omega_1^2\omega_2^2\omega_3^2+2\omega_1^3\omega_2^2\omega_3\right)d\omega_1\,d\omega_2\,d\omega_3$
	α_{013}	$-3\frac{\partial^7}{\partial\tau_1^3\partial\tau_2^2\partial\tau_3^2}$	$\frac{3i}{8}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi_{xxx}\omega_1^3\omega_2^2\omega_2^2\,d\omega_1\,d\omega_2\,d\omega_3$
	α_{004}	$3\left(\frac{\partial^8\psi_{xxx}}{\partial\tau_1^4\partial\tau_2^2\partial\tau_3^2} + \frac{\partial^8\psi_{xxx}}{\partial\tau_1^3\partial\tau_2^3\partial\tau_3^2}\right)$	$\frac{3}{8}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi_{xxx}\left(\omega_1^4\omega_2^2\omega_3^2+\omega_1^3\omega_2^3\omega_3^2\right)d\omega_1\,d\omega_2\,d\omega_3$

[a] The functions $\psi_x(\tau)$, $\psi_{xx}(\tau_1,\tau_2)$, and $\psi_{xxx}(\tau_1,\tau_2,\tau_3)$ and their derivatives are to be evaluated at $\tau = \tau_1 = \tau_2 = \tau_3 = 0$.

TABLE 2.- THE FUNCTIONS $I_{ijk}(x_0)$, $J_{ijk}(x_0)$, AND $J_{ijk}(-\infty)$

ijk	$e^{\frac{M_{00}}{2}x_0^2} I_{ijk}(x_0)$	$e^{\frac{M_{00}}{2}x_0^2} J_{ijk}(x_0)$	$J_{ijk}(-\infty)$
000	$\frac{1}{M_{22}}\left[1 + \xi e^{\frac{\xi^2}{2}} E(\xi)\right]$	$\frac{1}{M_{02}\sqrt{M_{22}}\alpha^2}\left[e^{\frac{\xi^2}{2}} E(\xi) + \sqrt{1+\alpha^2}\, e^{\frac{(1+\alpha^2)\xi^2}{2}} E^*\left(\sqrt{1+\alpha^2}\,\xi\right)\right]$	$\frac{\sqrt{2\pi}\sqrt{1+\alpha^2}}{M_{02}\sqrt{M_{22}}\alpha^2}$
300	$-\frac{M_{02}^3}{M_{22}^{3/2}}\left[\left\{\left(1 + 3\alpha^2 - 3\alpha^4\right)\xi + \alpha^6\xi^3\right\} + \alpha^2\left\{3 - 6\alpha^2\xi^2 + \alpha^4\xi^4\right\}e^{\frac{\xi^2}{2}} E(\xi)\right]$	$\frac{M_{02}^2}{M_{22}^2}\left[\left\{\left(-1 + \alpha^2\right) - \alpha^4\xi^2\right\} + \alpha^2\left\{3\xi - \alpha^2\xi\right\}e^{\frac{\xi^2}{2}} E(\xi)\right]$	0
201	$-\frac{M_{02}^2}{M_{22}^{3/2}}\left[\left(1 + 2\alpha^2\right)\xi + \alpha^2\left(1 - \alpha^2\xi^2\right)e^{\frac{\xi^2}{2}} E(\xi)\right]$	$\frac{M_{02}}{M_{22}}\left[-1 + \alpha^2\xi\, e^{\frac{\xi^2}{2}} E(\xi)\right]$	0
120	$-\frac{M_{11}M_{02}}{M_{22}^{3/2}}\left[-\alpha^2\xi + \left(1 - \alpha^2\xi^2\right)e^{\frac{\xi^2}{2}} E(\xi)\right]$	$\frac{M_{11}}{M_{22}}\left[1 + \xi\, e^{\frac{\xi^2}{2}} E(\xi)\right]$	0
102	$-\frac{M_{00}M_{22}^{1/2}}{M_{02}}\xi$	-1	0
003	$-M_{22}^{1/2}\xi$	$-\frac{M_{02}}{M_{00}}$	0
400	$\frac{M_{02}^4}{M_{22}^3}\left[\left\{-\left(1 + 6\alpha^2 - 3\alpha^4\right) + \left(1 + 4\alpha^2 + 6\alpha^4 - 6\alpha^6\right)\xi^2 + \alpha^8\xi^4\right\} + \alpha^4\left\{15\xi - 10\alpha^2\xi^3 + \alpha^4\xi^5\right\}e^{\frac{\xi^2}{2}} E(\xi)\right]$	$\frac{M_{02}^3}{M_{22}^{5/2}}\left[\left\{\left(1 + 3\alpha^2 - 3\alpha^4\right)\xi + \alpha^6\xi^3\right\} + \alpha^2\left\{3 - 6\alpha^2\xi^2 + \alpha^4\xi^4\right\}e^{\frac{\xi^2}{2}} E(\xi)\right]$	0
301	$\frac{M_{02}^3}{M_{22}^2}\left[\left\{-\left(1 + 3\alpha^2\right) + \left(1 + 3\alpha^2 + 3\alpha^4\right)\xi^2\right\} + \alpha^4\left\{3\xi - \alpha^2\xi^3\right\}e^{\frac{\xi^2}{2}} E(\xi)\right]$	$\frac{M_{02}^2}{M_{22}^{3/2}}\left[\left(1 + 2\alpha^2\right)\xi + \alpha^2\left(1 - \alpha^2\xi^2\right)e^{\frac{\xi^2}{2}} E(\xi)\right]$	0
220	$\frac{M_{11}M_{02}^2}{M_{22}^2}\left[-\left\{\left(1 - \alpha^2\right) + \alpha^4\xi^2\right\} + \alpha^2\left\{3\xi - \alpha^2\xi^3\right\}e^{\frac{\xi^2}{2}} E(\xi)\right]$	$\frac{M_{11}M_{02}}{M_{22}^{3/2}}\left[-\alpha^2\xi + \left(1 - \alpha^2\xi^2\right)e^{\frac{\xi^2}{2}} E(\xi)\right]$	0
202	$M_{00}\left[-1 + \left(1 + \alpha^2\right)\xi^2\right]$	$\frac{M_{00}M_{22}^{1/2}}{M_{02}}\xi$	0
121	$\frac{M_{11}M_{02}}{M_{22}}\left[-1 + \alpha^2\xi\, e^{\frac{\xi^2}{2}} E(\xi)\right]$	$\frac{M_{11}}{\sqrt{M_{22}}} e^{\frac{\xi^2}{2}} E(\xi)$	0
103	$M_{02}\left[-1 + \left(1 + \alpha^2\right)\xi^2\right]$	$M_{22}^{1/2}\xi$	0
040	$\frac{3M_{11}^2}{M_{22}}\left[1 + \xi\, e^{\frac{\xi^2}{2}} E(\xi)\right]$	$\frac{3M_{11}}{M_{02}\sqrt{M_{22}}\alpha^2}\left[e^{\frac{\xi^2}{2}} E(\xi) + \sqrt{1+\alpha^2}\, e^{\frac{(1+\alpha^2)\xi^2}{2}} E^*\left(\sqrt{1+\alpha^2}\,\xi\right)\right]$	$\frac{3\sqrt{2\pi}M_{11}^2\sqrt{1+\alpha^2}}{M_{02}\sqrt{M_{22}}\alpha^2}$
022	$-M_{11}$	$-\frac{M_{11}}{\sqrt{M_{00}}} e^{\frac{1+\alpha^2}{2}\xi^2} E^*\left(\sqrt{1+\alpha^2}\,\xi\right)$	$-\frac{\sqrt{2\pi}M_{11}}{\sqrt{M_{00}}}$
004	$M_{22}\left(-1 + \xi^2\right)$	$-\frac{M_{02}^2}{M_{00}^{3/2}}\left[-\sqrt{1+\alpha^2}\,\xi + \alpha^2 e^{\frac{1+\alpha^2}{2}\xi^2} E^*\left(\sqrt{1+\alpha^2}\,\xi\right)\right]$	$-\frac{\sqrt{2\pi}M_{02}^2\alpha^2}{M_{00}^{3/2}}$

TABLE 2.- THE FUNCTIONS $I_{ijk}(x_0)$, $J_{ijk}(x_0)$, AND $J_{ijk}(-\infty)$ - Concluded

ijk	$e^{\frac{M_{00}}{2}x_0^2} I_{ijk}(x_0)$	$e^{\frac{M_{00}}{2}x_0^2} J_{ijk}(x_0)$	$J_{ijk}(-\infty)$
600	$\frac{M_{02}^6}{M_{22}^4}\left[\left\{(3 + 15\alpha^2 + 45\alpha^4 - 15\alpha^6) - (6 + 33\alpha^2 + 75\alpha^4 + 90\alpha^6 - 45\alpha^8)\xi^2 + (1 + 6\alpha^2 + 15\alpha^4 + 20\alpha^6 + 15\alpha^8 - 15\alpha^{10})\xi^4 + \alpha^{12}\xi^6\right\} + \alpha^6\left\{-105\xi + 105\alpha^2\xi^3 - 21\alpha^4\xi^5 + \alpha^6\xi^7\right\}e^{\frac{\xi^2}{2}} E(\xi)\right]$	$\frac{M_{02}^5}{M_{22}^{7/2}}\left[\left\{(3 + 15\alpha^2 + 30\alpha^4 - 15\alpha^6)\xi - (1 + 5\alpha^2 + 10\alpha^4 + 10\alpha^6 - 10\alpha^8)\xi^3 - \alpha^{10}\xi^5\right\} + \alpha^4\left\{15 - 45\alpha^2\xi^2 + 15\alpha^4\xi^4 - \alpha^6\xi^6\right\}e^{\frac{\xi^2}{2}} E(\xi)\right]$	0
501	$\frac{M_{02}^5}{M_{22}^3}\left[\left\{(3 + 10\alpha^2 + 15\alpha^4) - (6 + 25\alpha^2 + 40\alpha^4 + 30\alpha^6)\xi^2 + (1 + 5\alpha^2 + 10\alpha^4 + 10\alpha^6 + 5\alpha^8)\xi^4\right\} + \alpha^6\left\{-15\xi + 10\alpha^2\xi^3 - \alpha^4\xi^5\right\}e^{\frac{\xi^2}{2}} E(\xi)\right]$	$-\frac{M_{02}^4}{M_{22}^{5/2}}\left[\left\{(3 + 10\alpha^2 + 12\alpha^4)\xi - (1 + 4\alpha^2 + 6\alpha^4 + 4\alpha^6)\xi^3\right\} + \alpha^4\left\{3 - 6\alpha^2\xi^2 + \alpha^4\xi^4\right\}e^{\frac{\xi^2}{2}} E(\xi)\right]$	0
420	$-\frac{M_{11}M_{02}^4}{M_{22}^3}\left[\left\{-(1 + 6\alpha^2 - 3\alpha^4) + (1 + 4\alpha^2 + 6\alpha^4 - 6\alpha^6)\xi^2 + \alpha^8\xi^4\right\} + \alpha^4\left\{15\xi - 10\alpha^2\xi^3 + \alpha^4\xi^5\right\}e^{\frac{\xi^2}{2}} E(\xi)\right]$	$-\frac{M_{11}M_{02}^3}{M_{22}^{5/2}}\left[\left\{(1 + 3\alpha^2 - 3\alpha^4)\xi + \alpha^6\xi^3\right\} + \alpha^2\left\{3 - 6\alpha^2\xi^2 + \alpha^4\xi^4\right\}e^{\frac{\xi^2}{2}} E(\xi)\right]$	0
402	$M_{00}^2\left[3 - 6(1 + \alpha^2)\xi^2 + (1 + \alpha^2)^2\xi^4\right]$	$-M_{00}^{3/2}\sqrt{1 + \alpha^2}\left[3\xi - (1 + \alpha^2)\xi^3\right]$	0
321	$-\frac{M_{11}M_{02}^3}{M_{22}^2}\left[\left\{-(1 + 3\alpha^2) + (1 + 3\alpha^2 + 3\alpha^4)\xi^2\right\} + \alpha^4\left\{3\xi - \alpha^2\xi^3\right\}e^{\frac{\xi^2}{2}} E(\xi)\right]$	$-\frac{M_{11}M_{02}^2}{M_{22}^{3/2}}\left[(1 + 2\alpha^2)\xi + \alpha^2(1 - \alpha^2\xi^2)e^{\frac{\xi^2}{2}} E(\xi)\right]$	0
303	$M_{00}M_{02}\left[3 - 6(1 + \alpha^2)\xi^2 + (1 + \alpha^2)^2\xi^4\right]$	$-M_{00}\sqrt{M_{22}}\left[3\xi - (1 + \alpha^2)\xi^3\right]$	0
240	$\frac{3M_{11}^2M_{02}^2}{M_{22}^2}\left[\left\{(1 - \alpha^2) + \alpha^4\xi^2\right\} + \alpha^2\left\{-3\xi + \alpha^2\xi^3\right\}e^{\frac{\xi^2}{2}} E(\xi)\right]$	$-\frac{3M_{11}^2M_{02}}{M_{22}^{3/2}}\left[-\alpha^2\xi + (1 - \alpha^2\xi^2)e^{\frac{\xi^2}{2}} E(\xi)\right]$	0
222	$M_{00}M_{11}\left[1 - (1 + \alpha^2)\xi^2\right]$	$-\frac{M_{11}M_{00}M_{22}^{1/2}}{M_{02}}\xi$	0
204	$M_{02}^2\left[(3 + \alpha^2) - (6 + 7\alpha^2 + \alpha^4)\xi^2 + (1 + \alpha^2)^2\xi^4\right]$	$-M_{02}\sqrt{M_{22}}\left\{(3 + \alpha^2)\xi - (1 + \alpha^2)\xi^3\right\}$	0
141	$\frac{3M_{11}^2M_{02}}{M_{22}}\left[1 - \alpha^2\xi e^{\frac{\xi^2}{2}} E(\xi)\right]$	$-\frac{3M_{11}^2}{\sqrt{M_{22}}}e^{\frac{\xi^2}{2}} E(\xi)$	0
123	$M_{11}M_{02}\left[1 - (1 + \alpha^2)\xi^2\right]$	$-M_{11}M_{22}^{1/2}\xi$	0
105	$M_{02}M_{22}\left[3 - 3(2 + \alpha^2)\xi^2 + (1 + \alpha^2)\xi^4\right]$	$-M_{22}^{3/2}\left[3\xi - \xi^3\right]$	0
060	$-\frac{15M_{11}^3}{M_{22}}\left[1 + \xi e^{\frac{\xi^2}{2}} E(\xi)\right]$	$-\frac{15M_{11}^3M_{02}^4}{M_{00}^{5/2}}\left[e^{\frac{\xi^2}{2}} E(\xi) + \sqrt{1 + \alpha^2}\, e^{\frac{(1+\alpha^2)\xi^2}{2}} E^*\left(\sqrt{1 + \alpha^2}\,\xi\right)\right]$	$-\frac{15\sqrt{2\pi}M_{11}^3\sqrt{1 + \alpha^2}}{M_{02}\sqrt{M_{22}}\alpha^2}$
042	$3M_{11}^2$	$\frac{3M_{11}^2}{\sqrt{M_{00}}}e^{\frac{1+\alpha^2}{2}\xi^2} E^*\left(\sqrt{1 + \alpha^2}\,\xi\right)$	$\frac{3\sqrt{2\pi}M_{11}^2}{\sqrt{M_{00}}}$
024	$M_{11}M_{22}(1 - \xi^2)$	$\frac{M_{11}M_{02}^2}{M_{00}^{3/2}}\left[-\sqrt{1 + \alpha^2}\,\xi + \alpha^2 e^{\frac{1+\alpha^2}{2}\xi^2} E^*\left(\sqrt{1 + \alpha^2}\,\xi\right)\right]$	$\frac{\sqrt{2\pi}M_{11}M_{02}^2\alpha^2}{M_{00}^{3/2}}$
006	$M_{22}^2(3 - 6\xi^2 + \xi^4)$	$\frac{M_{02}^4}{M_{00}^{5/2}}\left[\left\{-(3 + 6\alpha^2)\sqrt{1 + \alpha^2}\,\xi + (1 + \alpha^2)^{3/2}\xi^3\right\} + 3\alpha^4 e^{\frac{1+\alpha^2}{2}\xi^2} E^*\left(\sqrt{1 + \alpha^2}\,\xi\right)\right]$	$\frac{3\sqrt{2\pi}M_{02}^4\alpha^4}{M_{00}^{5/2}}$